Perturbation theory

And its observables

LECTURE 1

Chapter 1. Homogeneous Universe

How do we describe the Universe?

- Depends on what we see:
  - Galaxies and groups
  - A Cosmic background radiation
  - Distance tracers

- Depends on what we are looking for:
  - The origin of the Universe (as a whole)
  - Why the universe is in an accelerated expansion?
  - **How structures assemble?**

The unperturbed Universe

- Cosmological principle estipulates that universe is invariant:

Wherever you stand **(homogeneous)** and whichever direction you look at **(isotropic)**

- Comoving coordinates and uniform dynamics yield,

$$\underline{R}(t,r) = a(t)\underline{r} \quad \longrightarrow \quad \boxed{\underline{v} = H\underline{R}(t,r)} \quad \text{where} \quad H \equiv \frac{1}{a}\frac{d}{dt}a = \frac{\dot{a}}{a}$$

:

The expanding Universe *by E.A. Poe*

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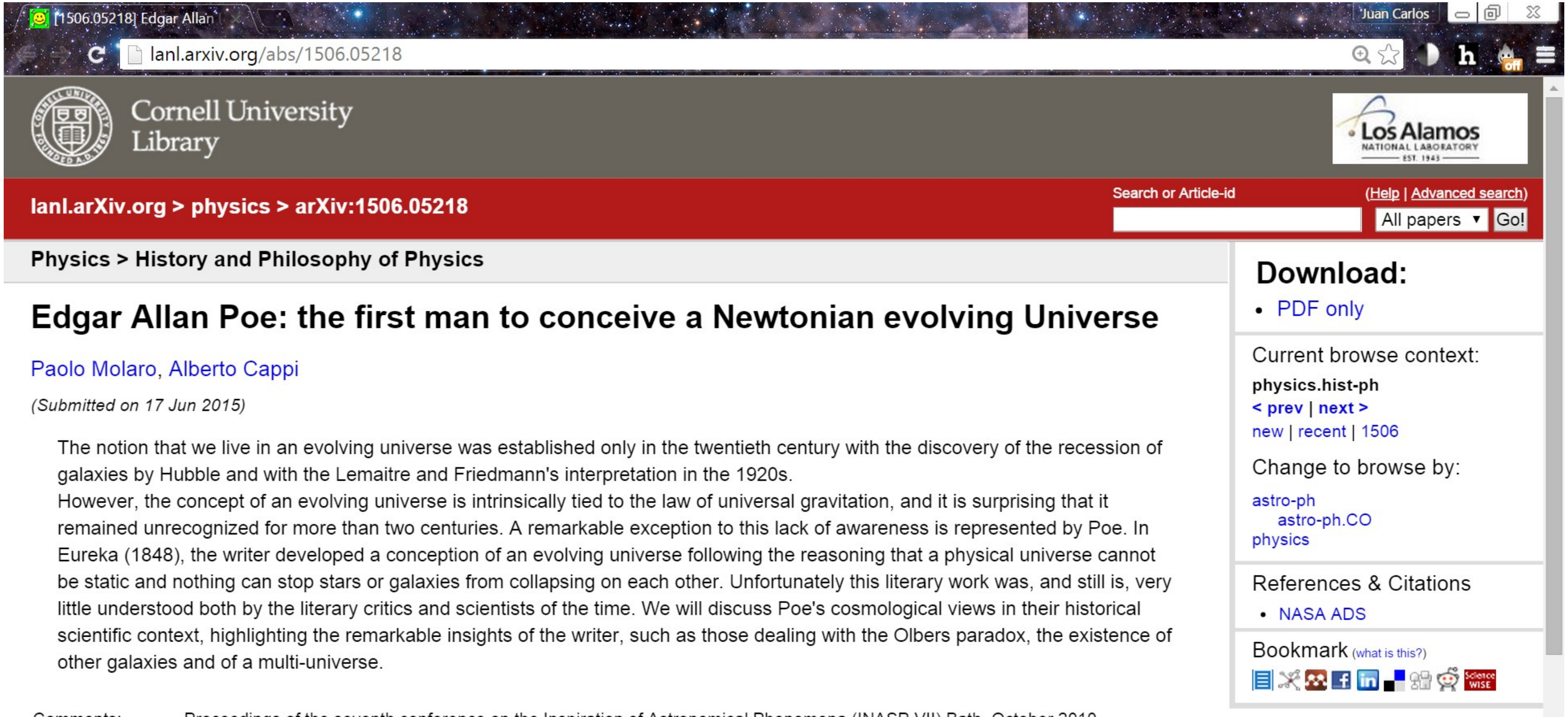
[1506.05218] Edgar Allan

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Physics > History and Philosophy of Physics

Edgar Allan Poe: the first man to conceive a Newtonian evolving Universe

Paolo Molaro, Alberto Cappi

(Submitted on 17 Jun 2015)

The notion that we live in an evolving universe was established only in the twentieth century with the discovery of the recession of galaxies by Hubble and with the Lemaitre and Friedmann's interpretation in the 1920s. However, the concept of an evolving universe is intrinsically tied to the law of universal gravitation, and it is surprising that it remained unrecognized for more than two centuries. A remarkable exception to this lack of awareness is represented by Poe. In Eureka (1848), the writer developed a conception of an evolving universe following the reasoning that a physical universe cannot be static and nothing can stop stars or galaxies from collapsing on each other. Unfortunately this literary work was, and still is, very little understood both by the literary critics and scientists of the time. We will discuss Poe's cosmological views in their historical scientific context, highlighting the remarkable insights of the writer, such as those dealing with the Olbers paradox, the existence of other galaxies and of a multi-universe.

Comments: Proceedings of the seventh conference on the Inspiration of Astronomical Phenomena (INSAP VII) Bath, October 2010

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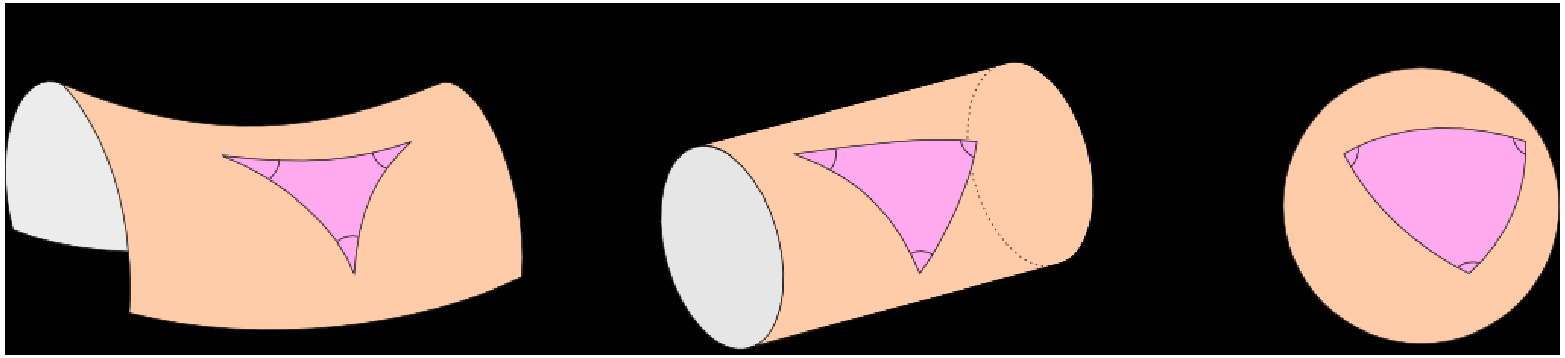
- RW proved that the metric is the most general case of an homogeneous and isotropic expansion

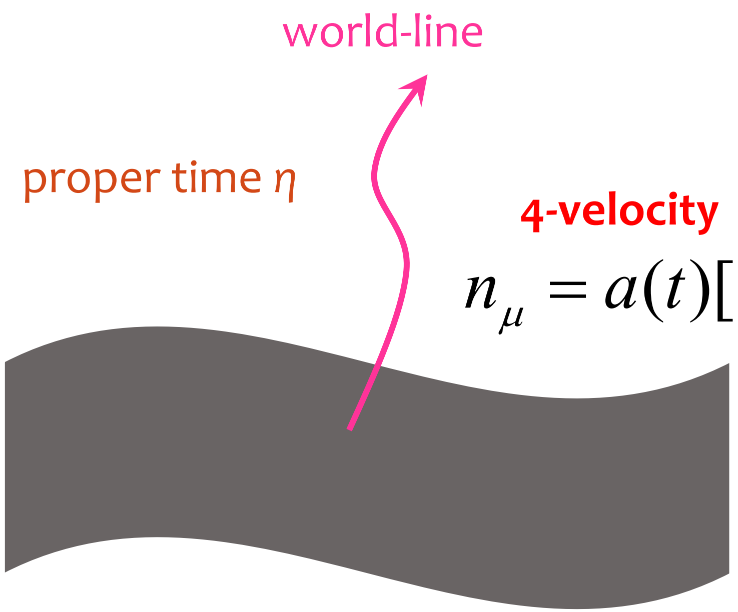
$$ds^2 = a(\eta)^2 \left[-d\eta^2 + \frac{dr^2}{1-kr^2} + r^2 (d\theta^2 + \sin(\theta)^2 d\phi^2) \right]$$

Where spatial curvature *k* defines the geometry



Observers and kinematics

- Comoving (orthogonal) observer, $$n_\mu = \frac{d\eta}{dx^\mu} \qquad n_\mu n^\mu = -1$$
- Kinematic quantities defined with projection tensor $$P_{\nu\mu} \equiv g_{\nu\mu} + n_\mu n_\nu$$



4-velocity

$$n_\mu = a(t)[-1, 0, 0, 0]$$

$$P_{\nu\mu} \equiv g_{\nu\mu} + n_\mu n_\nu = a(t)[\eta_{\mu\nu} + \delta_{\mu 0}\delta_{0\nu}]$$

projector on Σ

Observers and kinematics

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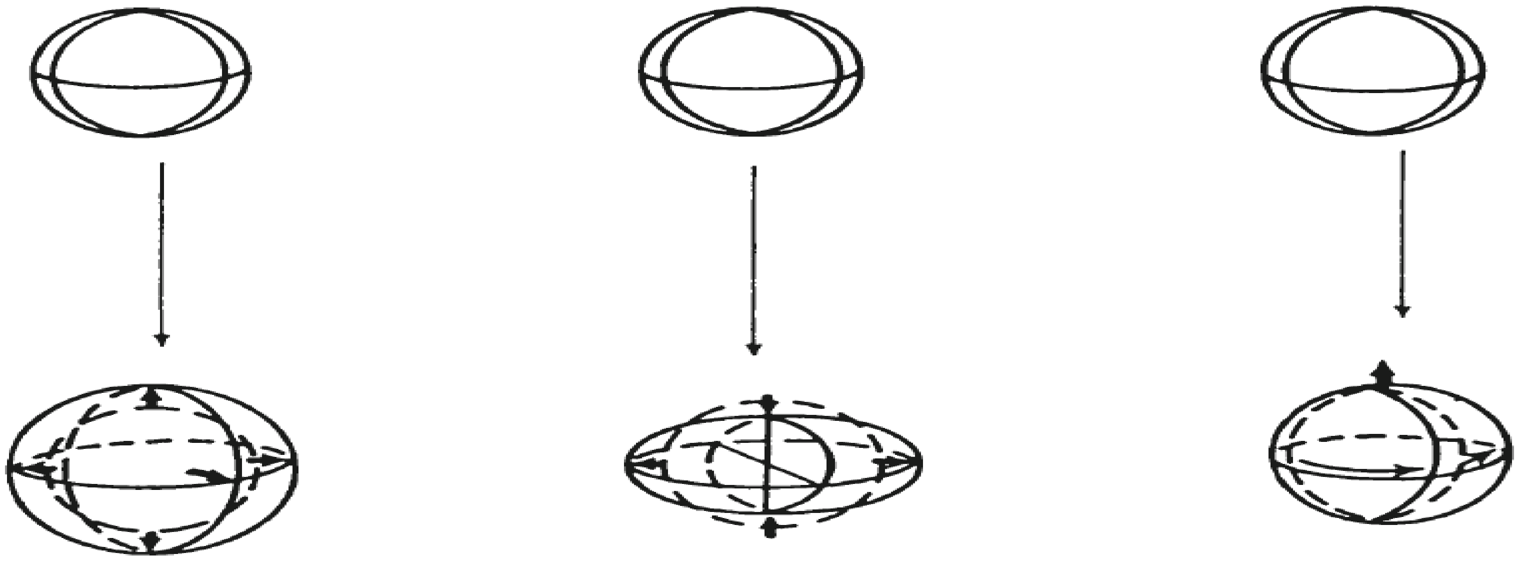
$$n_{\mu;\nu} = \frac{1}{3}\theta\,\mathcal{P}_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu} - a_\mu n_\nu \,,$$

- Expansion θ, vorticity $\omega_{\mu\nu}$, shear $\sigma_{\mu\nu}$, and acceleration a_μ.

$$\theta = n^\mu{}_{;\mu}\,, \qquad \sigma_{\mu\nu} = \frac{1}{2}\mathcal{P}_\mu{}^\alpha\mathcal{P}_\nu{}^\beta\left(n_{\alpha;\beta} + n_{\beta;\alpha}\right) - \frac{1}{3}\theta\mathcal{P}_{\mu\nu}\,, \qquad \omega_{\mu\nu} = \frac{1}{2}\mathcal{P}_\mu{}^\alpha\mathcal{P}_\nu{}^\beta\left(n_{\alpha;\beta} - n_{\beta;\alpha}\right)\,,$$



Geometrical quantities

- The rate of change of an infinitesimally commoving volume V is given by $\frac{1}{V}\frac{dV}{d\eta} = \theta = 3H$
- The Lie derivative of the projection tensor along the velocity field is the extrinsic curvature of spatial hypersurfaces

$$K_{\mu\nu} \equiv \frac{1}{2}\pounds_n \mathcal{P}_{\mu\nu} = \mathcal{P}_\nu{}^\lambda n_{\mu;\lambda} = \frac{1}{3}\theta\,\mathcal{P}_{\mu\nu} + \sigma_{\mu\nu}\,. \qquad K^\mu{}_\mu = K = H$$

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- In an FLRW universe, the comoving horizon is related to the (comoving) Hubble radius $r_H=\frac{c}{aH}=\frac{c}{\mathcal{H}}$
  - From

$$ds^2=0\rightarrow d\eta=dr$$

$$r_c=\int_0^{\eta}d\eta'=\int_0^{a}\frac{c}{aH}d\log a=\int_0^{a}r_H d\log a=\begin{cases}r_H & \text{Radiation domination}\\ 2r_H & \text{Matter domination}\end{cases}$$

thus Hubble Horizon $r_H=\frac{c}{aH}=\frac{c}{\mathcal{H}}$

Unperturbed ingredients

- Stress Energy tensor with anisotropic stress.

$$T^{\mu}{}_{\nu} = (\rho + p)\, u^{\mu} u_{\nu} + p\delta^{\mu}{}_{\nu} + \pi^{\mu}{}_{\nu}\,, \qquad T^{\mu}{}_{\nu} u^{\nu} = -\rho u^{\mu}\,.$$

- Proper frame of perfect fluid $T_{\mu\nu} = \mathrm{diag}(-\rho, p, p, p).$ (with $p = \omega\rho$)
- Dark matter presents no velocity dispersion, no pressure, no shear .

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- From distribution function

$$T^{\mu\nu} = \frac{g}{(2\pi)^3} \int f p^{\mu} p^{\nu} \frac{d^3p}{E}$$

- Energy density, momentum density and pressure

$$\rho = \frac{g}{(2\pi)^3} \int f E\, d^3p \qquad (\rho + P)\mathbf{v} = \frac{g}{(2\pi)^3} \int f \mathbf{p}\, d^3p \qquad P = \frac{g}{3(2\pi)^3} \int f p^2 \frac{d^3p}{E}$$

- For ultra-relativistic particles $E = P$, thus $\rho = 3P$.
- also in equilibrium this is a perfect fluid

Unperturbed ingredients

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- Proper frame of perfect fluid $T_{\mu\nu} = \mathrm{diag}(-\rho, p, p, p).$ (with $p = \omega\rho$)
- Dark matter presents no velocity dispersion, no pressure, no shear .
- A scalar field has

$$T^{\mu}{}_{\nu} = g^{\mu\alpha}\varphi_{,\alpha}\varphi_{,\nu} - \delta^{\mu}{}_{\nu}\left(U(\varphi) + \frac{1}{2} g^{\kappa\lambda}\varphi_{,\kappa}\varphi_{,\lambda}\right)$$

- The homogeneous field can be described as a perfect fluid

$$u_{\mu} = \frac{\varphi_{,\mu}}{|g^{\lambda\kappa}\varphi_{,\lambda}\varphi_{,\kappa}|} \qquad \rho = -g^{\lambda\kappa}\varphi_{,\lambda}\varphi_{,\kappa} + U\,, \quad P = -g^{\lambda\kappa}\varphi_{,\lambda}\varphi_{,\kappa} - U$$

Evolution of horizons in FLRW

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$

- Einstein equations (and Conservation of $T_{\mu\nu}$) for a perfect fluid ($p = \omega\rho$)

$$\frac{d\rho}{dt} = 3H(P+\rho) \qquad H^2 = \frac{8\pi G}{3}\rho - \frac{Kc^2}{a^2} \qquad \Omega_m + \Omega_\Lambda + \Omega_\kappa = \left(\frac{H}{H_0}\right)^2$$

- Solutions

$$\rho = \rho_0 a^{-3(1+\omega)} \qquad a(t) = \left(\frac{t}{t_0}\right)^{2/3(\omega+1)}$$

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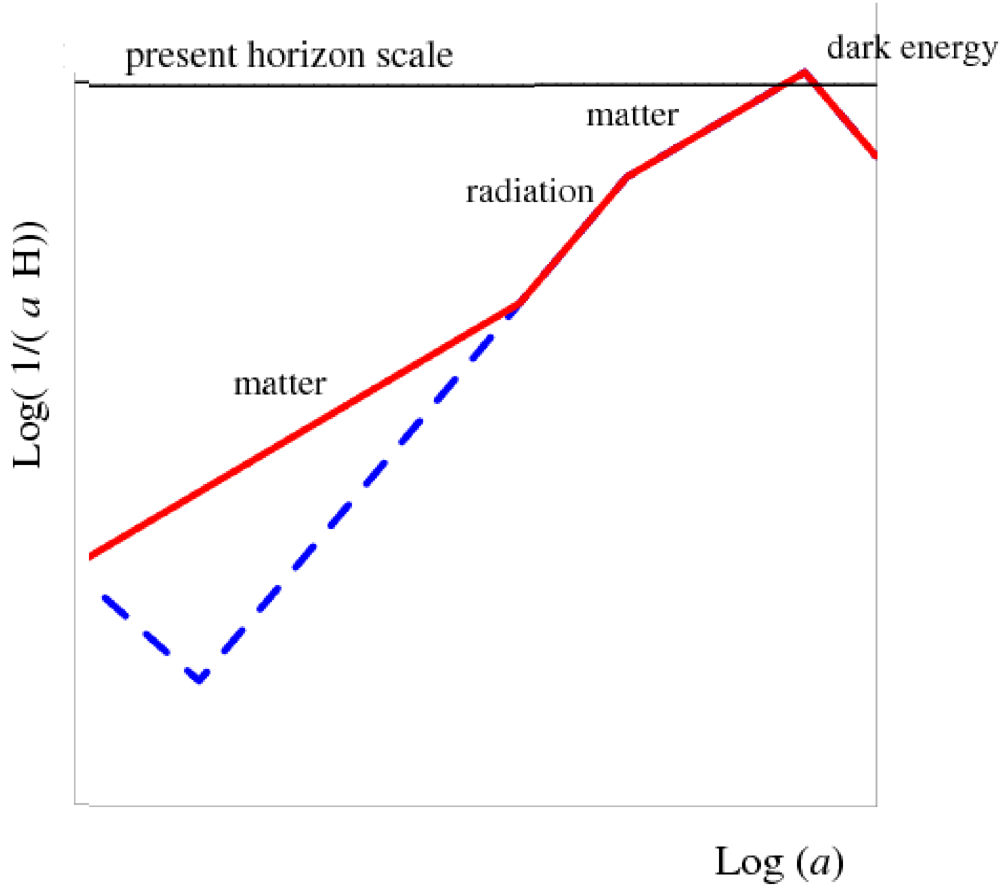
- Solutions

$$\rho = \rho_0 a^{-3(1+\omega)} \qquad a(t) = \left(\frac{t}{t_0}\right)^{2/3(\omega+1)}$$

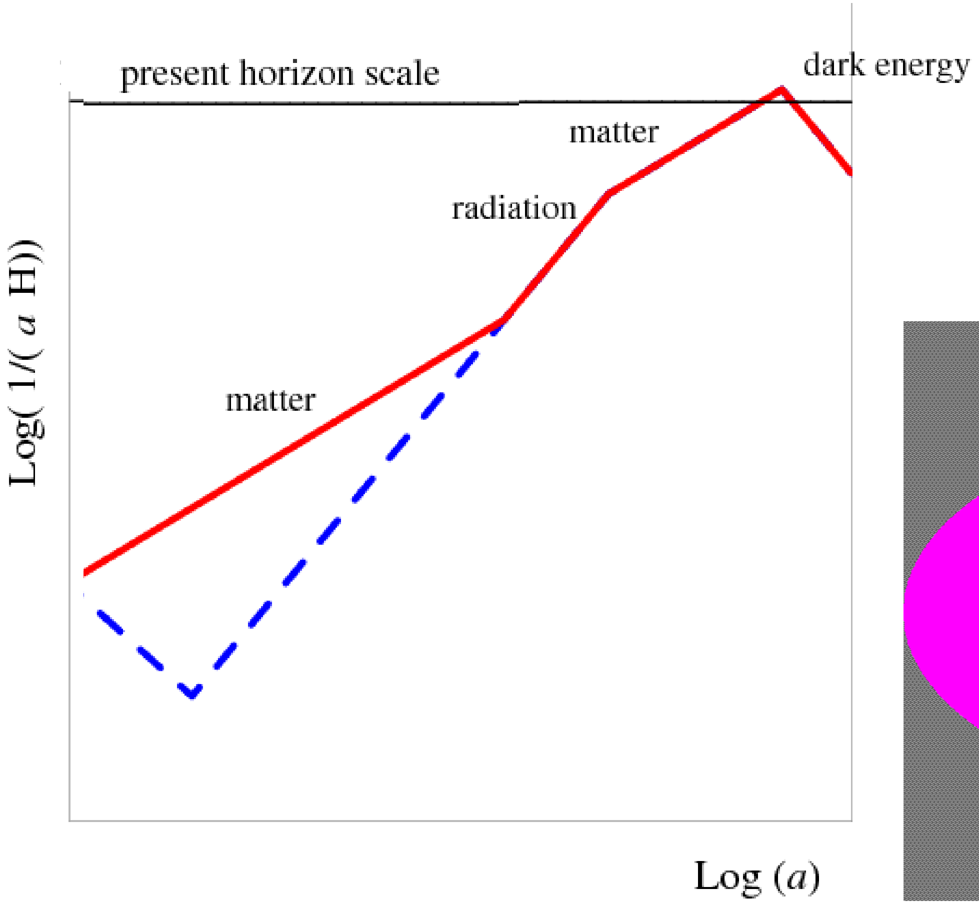
- Comoving horizon ever expanding

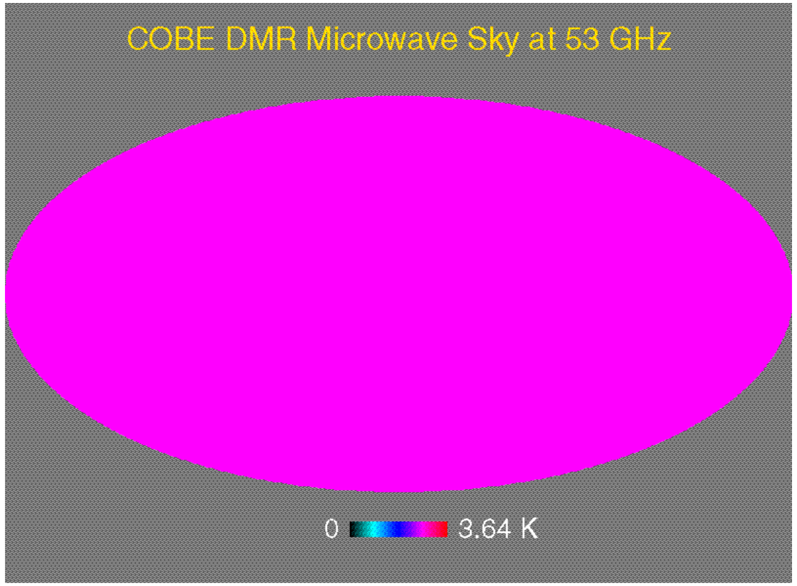


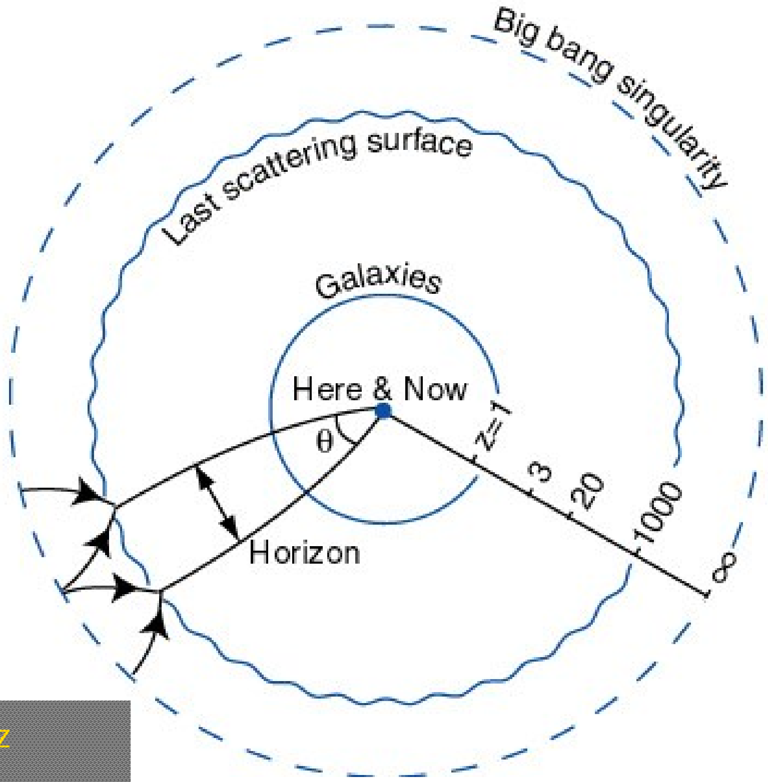
$$\frac{dr_H}{dt} = - \quad \tau G \quad (p + \rho/3)$$

(Problem for Big Bang)

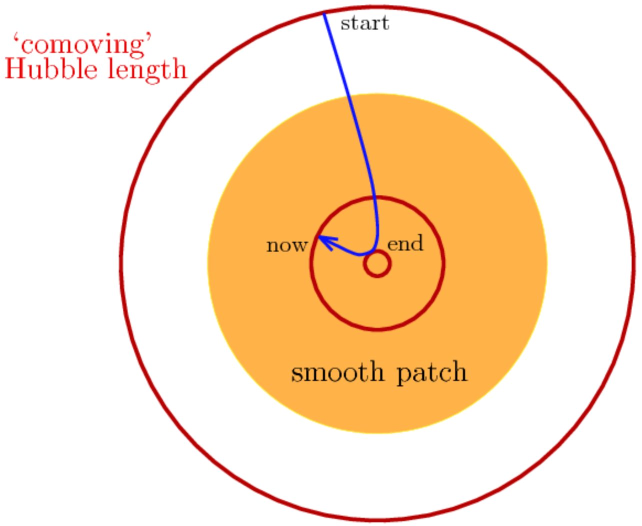
- Homogeneous Universe beyond the r_H at recombination.







Solution: Inflation



- Homogeneous Universe beyond the r_H at recombination.
- Require a shrinking r_H for early times: inflation

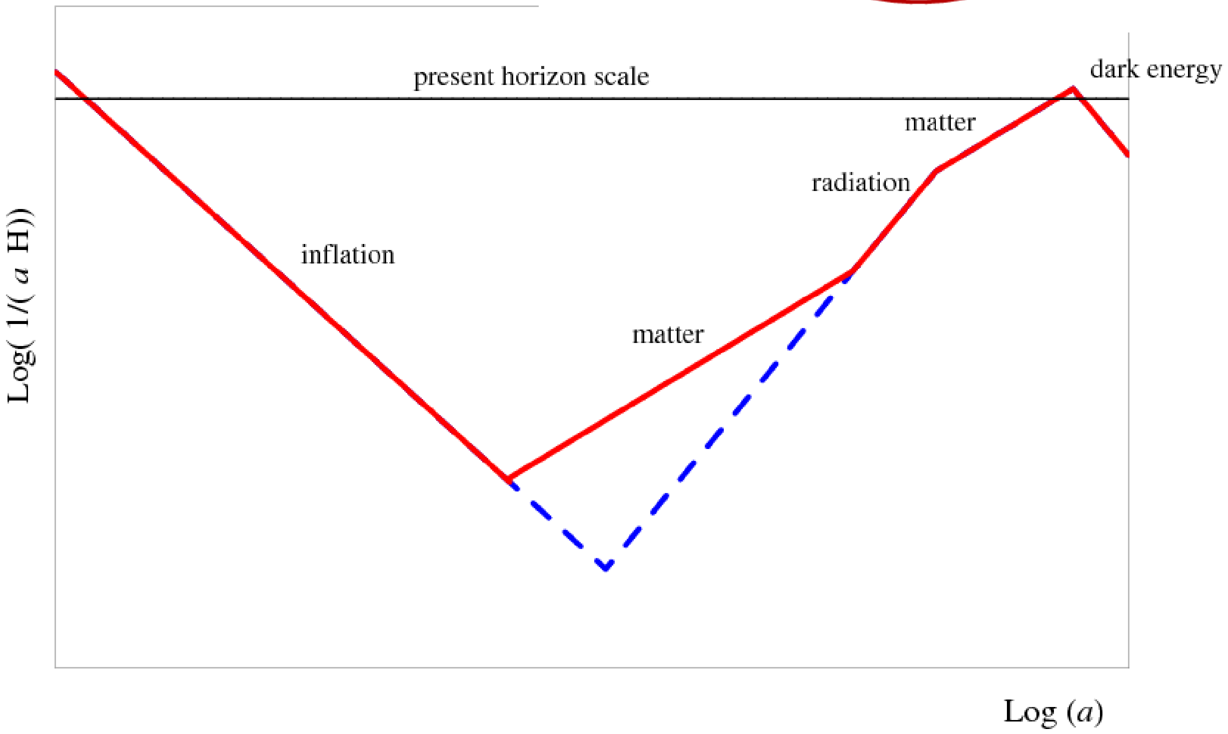
$$\frac{d}{dt}\left(\frac{1}{aH}\right) = -\frac{\ddot{a}}{\dot{a}^2} << 0 \qquad \frac{\ddot{a}}{a} = -4\pi G(p + \frac{1}{3}\rho)$$

- Scalar field (inflaton)

$$\rho = \frac{1}{2}(\dot{\varphi})^2 + V(\varphi), \quad p = \frac{1}{2}(\dot{\varphi})^2 - V(\varphi)$$

- Slow roll conditions.

$$\dot{\varphi}^2 << V(\varphi) \qquad \ddot{\varphi} << 3H\dot{\varphi}$$



Chapter 2. Perturbations

What are perturbations?

- Approximating scheme to solve problems from solutions to related simplifications. Example:

$$\sqrt{26} = \sqrt{25+1} = 5\sqrt{1+\frac{1}{25}} \approx 5(1+1/50) \approx 5.1(=5.099) \rightarrow \quad \sqrt{y} = \sqrt{x^2\left(1+\varepsilon\right)} = x\sqrt{1+\varepsilon}$$

For tensors $\quad \mathbf{T}(\eta, x^i) = \mathbf{T}_0(\eta) + \delta\mathbf{T}(\eta, x^i)\,.$

And a Taylor expansion

$$\delta\mathbf{T}(\eta, x^i) = \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!}\delta\mathbf{T}_n(\eta, x^i)\,,$$

What are perturbations?

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For tensors $\mathbf{T}(\eta, x^i) = \mathbf{T}_0(\eta) + \delta\mathbf{T}(\eta, x^i)\,.$

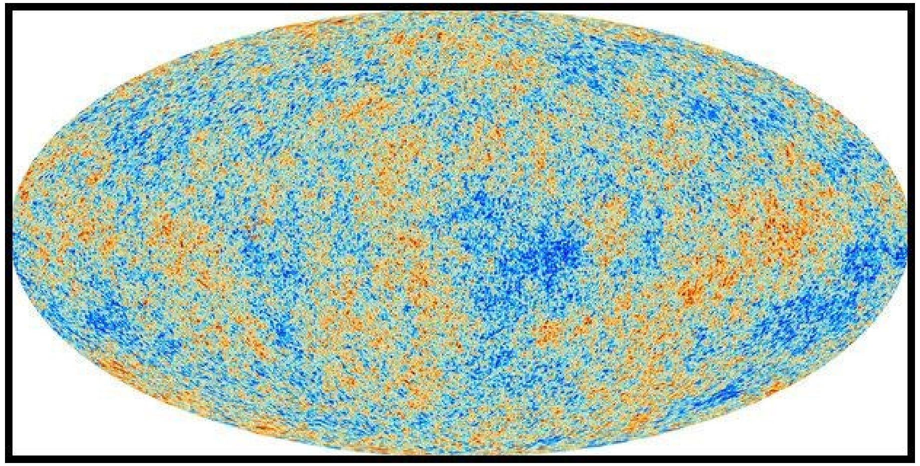
And a Taylor expansion $$\delta\mathbf{T}(\eta, x^i) = \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!}\delta\mathbf{T}_n(\eta, x^i)\,,$$

In Cosmological Perturbation Theory we deal with deviations from FLRW Universe:

- Inhomogeneities or anisotropies. $\rho(x,t) = \bar{\rho}(t) + \delta\rho(x,t) = \bar{\rho}(t)(1+\delta)$

- Why Perturbations? Observations from CMB:

$$\frac{\delta T}{T} \simeq$$



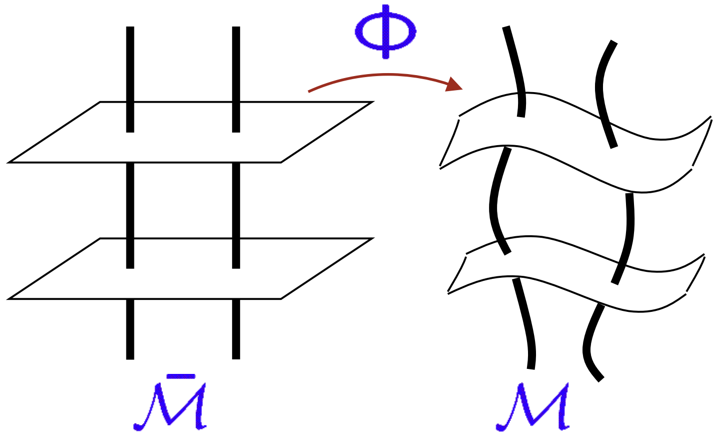
What are perturbations?

$$\rho(x,t) = \bar{\rho}(t) + \delta\rho(x,t) = \bar{\rho}(t)(1+\delta)$$

- Unambiguous definition of perturbations requires a map.

$$\delta\rho = \underbrace{\rho}_{\mathcal{M}} - \underbrace{\bar{\rho}}_{\bar{\mathcal{M}}} \ ?$$

$$\delta Q = Q - \Phi(\bar{Q})$$



- The map must account for a small deviation from **background.**
  - What if $\bar{Q} = 0$? Then δQ is independent of mapping.
  - Then δQ is **Gauge-Independent** (Stewart-Walker Lemma)

What are perturbations?

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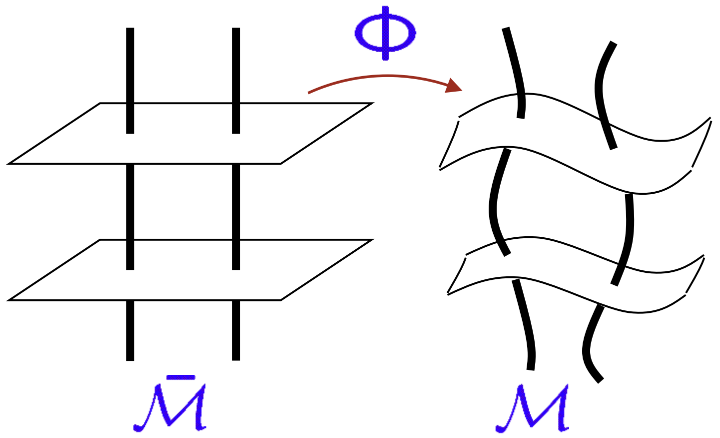

- The map must account for a small deviation from **background**.

- No unique way of defining this map Φ (re: coordinate choice).
- No unique way of defining perturbations (i.e. gradient expansion).

$$\nabla^2 Q/(aH)^2 \rightarrow -k^2 Q/(aH)^2 << 1 \quad \text{(at super-horizon scales)}$$

Metric Perturbations

- The metric tensor perturbations $g_{\mu\nu} = \bar{g}_{\mu\nu}(t) + \delta g_{\mu\nu}(x,t)$

$$\delta g_{00} = -2a^2\phi$$ → Gravitational potential

$$\delta g_{0i} = a^2(B_{,i} - S_i)$$

$$\delta g_{ij} = 2a^2(-2\psi + E_{,ij} + F_{i,j} + h_{ij})$$

Potential shift

Metric Perturbations

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Potential shift

$\delta g_k^{\ k}$ = Local scale factor

$(\partial_i\partial_j - \tfrac{1}{3}\nabla^2)(E' + B) =$ Shear scalar

- Split from Helmholtz Theorem
- Scalar, vector and tensors **decouple at first order.**

Metric Perturbations

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Potential shift

$\delta g_k^{\ k}$ = Local scale factor

$$(\partial_i\partial_j - \tfrac{1}{3}\nabla^2)(E + B) =$$ Shear scalar σ

- Geometrical quantities:
  - Curvature of spatial hypersurfaces $$^{(3)}R_1 = \frac{4}{a^2}\nabla^2\psi_1$$
  - Acceleration $$a_i = \phi_{,i}$$
  - Expansion
  - Proper time

$$d\tau = (1+\phi)dt$$

$$\theta = \frac{3}{a}\left[\mathcal{H} - \mathcal{H}\phi - \psi' + \frac{1}{3}\nabla^2\sigma\right]$$

$T_{\mu\nu}$ Perturbations

- The Stress-Energy tensor split $T_{\mu\nu} = \bar{T}_{\mu\nu}(t) + \delta T_{\mu\nu}(x,t)$

$$\delta T^0_{\ 0} = -\delta\rho_1 \,,$$

$$\delta T^0_{\ i} = (\rho_0 + P_0)(v_{1i} + B_{1i})$$

$$\delta T^i_{\ j} = \delta P_1 \delta^i_{\ j} + a^{-2}\pi^{\ \ i}_{(1)\ j} \,,$$

Matter density perturbation

Velocity potential

$\delta T_k^{\ k}$ = Local pressure

Anisotropic stress

- Split not explicitly shown but $u^\mu = a^{-1}\left(1-\phi, v^i + v_{,}^{\ i}\right)$
- Scalar field:

$$\delta T^0_{\ 0} = a^{-2}\bar{\varphi}'(\phi\bar{\varphi}' - \delta\varphi') - U_{,\varphi}\delta\varphi$$

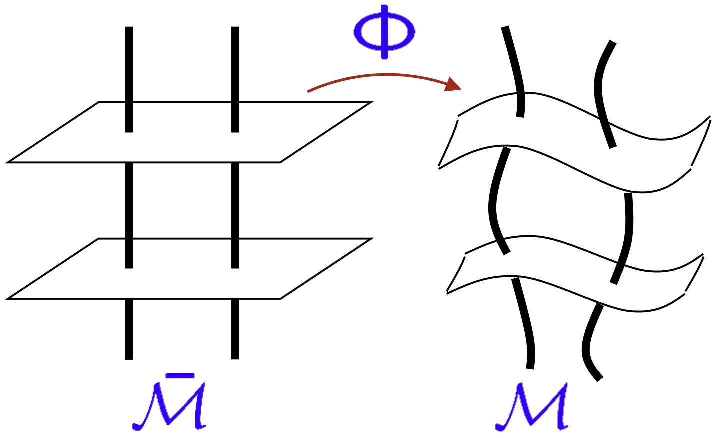
$$\delta T^0_{\ i} = -a^{-2}\bar{\varphi}'(\delta\varphi_{,i})$$

$$\delta T^i_{\ j} = \left[a^{-2}\bar{\varphi}'(\delta\varphi' - \phi\bar{\varphi}') - U_{,\varphi}\delta\varphi\right]\delta^i_{\ j}$$

Gauges

- $\bar{Q}(t)$ depends on our choice of equal-time hypersurface at each point (x,t) :
- So $\delta Q(r,t)$ will also depend on this choice of time-slicing or **gauge choice**

$$\delta Q = Q - \Phi(\bar{Q})$$



Gauge Transformation: $x^{\mu} \rightarrow x^{\mu} + \xi^{\mu}$
Coordinate transformation which map points of one slicing to another

1) Must be small change
2) Helmholtz split $\xi^{\mu} = (\alpha, \beta^{i} + \beta_{,}^{\ i})$
3) Imposed to specific characteristics of $\delta Q(r,t)$

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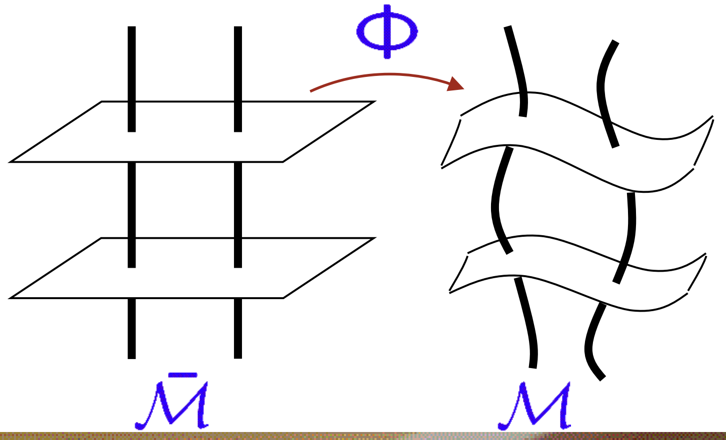
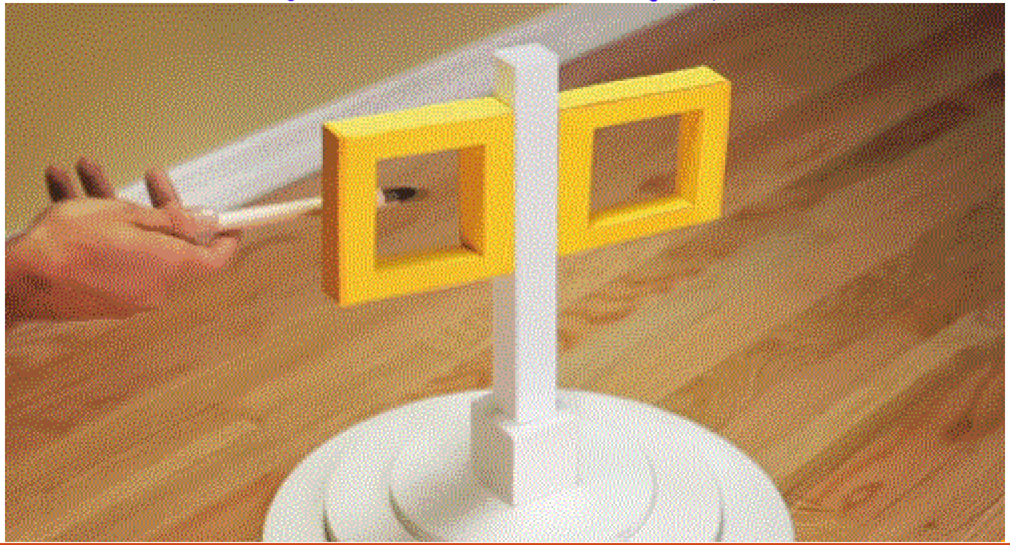


Gauge Transformation: $x^\mu \to x^\mu + \xi^\mu$

Coordinate transformation which map points of one slicing to another

1) Must be small change
2) Helmholtz split $\xi^\mu = (\alpha, \beta^i + \beta_{,}^{\ i})$
3) Imposed to specific characteristics of $\delta Q(r,t)$
4) OJO: spurious quantities may appear



Gauges

- Active approach: Map transforming perturbed quantities

$$\tilde{\mathbf{T}} = e^{\pounds_\xi}\mathbf{T}$$

- Vector field generating transformation $\xi^\mu = (\alpha, \beta^i + \beta_{,}^{i}) = \xi_1{}^\mu + \frac{1}{2}\xi_2{}^\mu$

- Expansion of exponential map $\exp(\pounds_\xi) = 1 + \epsilon\pounds_{\xi_1} + \frac{1}{2}\epsilon^2\pounds_{\xi_1}^2 + \frac{1}{2}\epsilon^2\pounds_{\xi_2} + \dots$

- Split of tensor transformation $\tilde{T}$

$$\tilde{T} \quad {}_1\bar{T}$$

$$\tilde{T} \quad {}_{\xi 2}\bar{T} + (\pounds_{\xi 1})^2\bar{T} + 2\pounds_{\xi 1}T_1$$

Gauges

$$\xi^{\mu} = (\alpha, \beta^{i} + \beta_{,}^{i}) = \varepsilon\xi_1^{\mu} + \tfrac{1}{2}\varepsilon^2\xi_2^{\mu}$$

- Passive approach: Provide relation between coordinates $\tilde{}$ and $x^{\mu}(q)$

$$\widetilde{x^{\mu}}(q) = x^{\mu}(q) - \epsilon\xi_1^{\mu}(q)$$

- Require total quantities invariant $\tilde{\rho}(\tilde{x}^{\mu}) = \rho(x^{\mu})$

- Expansion of both sides in perturbations $\rho(x^{\mu}) = \rho_0(x^0) + \epsilon\delta\rho_1(x^{\mu})$

$$\tilde{\rho}(\tilde{x}^{\mu}) = \rho_0\left(\widetilde{x^0}\right) + \epsilon\widetilde{\delta\rho_1}\left(\widetilde{x^{\mu}}\right) = \rho_0(x^0) + \epsilon\left(-\rho_0'(x^0)\xi_1^0(x^{\mu}) + \widetilde{\delta\rho_1}(x^{\mu})\right)$$

- Result: Transformation rule at first order:

$$\widetilde{\delta\rho_1} = \delta\rho_1 + \rho_0'\xi_1^0$$

- **Same applies for any other 4-scalar** $\tilde{\zeta}$ $\bar{\zeta}'\alpha_1$

Gauges

- Vector and tensor transformations computed through exponential map.
- Relevant results from vectors
  - Velocity transformation
  - Scalar off-diagonal metric $'-\alpha_1$
- Results from tensor transformations
  - Scalar metric potentials

$$\widetilde{\phi}_1 = \phi_1 + \mathcal{H}\alpha_1 + \alpha_1' \, ,$$

$$\widetilde{\psi}_1 = \psi_1 - \mathcal{H}\alpha_1 \, ,$$

$$\widetilde{E}_1 = E_1 + \beta_1 \, ,$$

  - Gravitational Waves

$$\boxed{\widetilde{h}_{1ij} = h_{1ij}}$$

Gauges…what Gauges?

- Observers may measure different observables
- Observers that see uniform field:
  - Require: $\delta\tilde{\rho}$ $\bar{\rho}'\alpha_1 = 0 \rightarrow \alpha_{1\rho}' = \frac{\delta\rho_1}{\bar{\rho}'}$
  - Result: Curvature perturbation in **uniform density gauge**.

$$\tilde{\psi} \quad \mathcal{H}\frac{\delta\rho_1}{\bar{\rho}'} \equiv -\zeta$$

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$$\tilde{\psi} \qquad \mathcal{H}\frac{\delta\rho_1}{\bar{\rho}'} \equiv -\zeta$$

- Observers with unperturbed spatial hypersurfaces:
  - Require $\tilde{\psi}$

$$\alpha_{\psi 1} = \psi_1 / \mathcal{H}, \;\; \beta_{\psi 1} = -E_1$$

  - Result: Field perturbation in **flat gauge**.

$$\delta\tilde{\varphi} \qquad \bar{\varphi}'\frac{\psi_1}{\mathcal{H}} \equiv Q_{MS}$$

Gauges…what Gauges?

- Observers may measure different observables
- Observers that see uniform scalar field:
  - Require:

$$\alpha_{\varphi 1}' = \frac{\delta\varphi_1}{\bar{\varphi}'}$$

  - Result: Curvature perturbation in **Uniform field gauge**.

$$\tilde{\psi} \qquad \mathcal{H}\frac{\delta\varphi_1}{\bar{\varphi}'}$$

Gauges…what Gauges?

- Observers may measure different observables
- Observers that see uniform scalar field:
  - Require:

$$\alpha_{\varphi 1}' = \frac{\delta\varphi_1}{\bar{\varphi}'}$$

  - Result: Curvature perturbation in **Uniform field gauge.**

$$\tilde{\psi} \qquad\qquad \mathcal{H}\frac{\delta\varphi_1}{\bar{\varphi}'}$$

- Observers that experience no shear:
  - Require

$$\alpha_{\ell} \qquad\qquad \sigma_1 = B_1 - E_1'$$

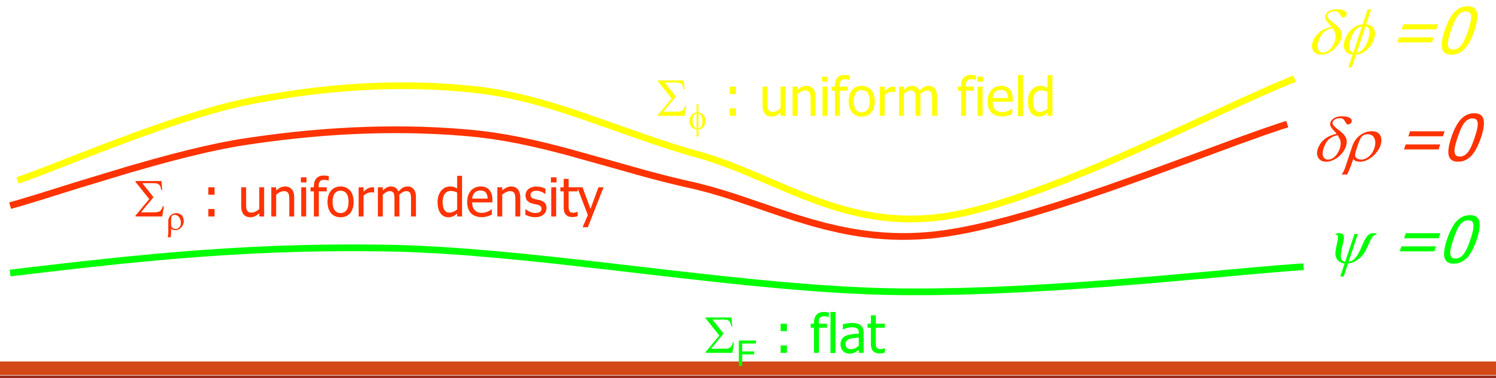
  - Result: Metric potentials in **Longitudinal or Newtonian Gauge.**

$$\tilde{\phi} \qquad\qquad \mathcal{H}\left(B_1 - E_1'\right) + \left(B_1 - E_1'\right)' \equiv \Phi$$

$$\tilde{\psi} \qquad\qquad -\mathcal{H}\left(B_1 - E_1'\right) \equiv \Psi$$

Gauge Invariants

- Combine two scalars $$\widetilde{\psi}_1 = \psi_1 - \mathcal{H}\alpha_1\,, \;\; \widetilde{\delta\rho_1} = \delta\rho_1 + \rho_0'\alpha_1$$
- Obtain a Gauge-invariant quantity from their difference $$\frac{\psi_1}{\mathcal{H}} - \frac{\delta\rho_1}{\bar{\rho}\,'}$$

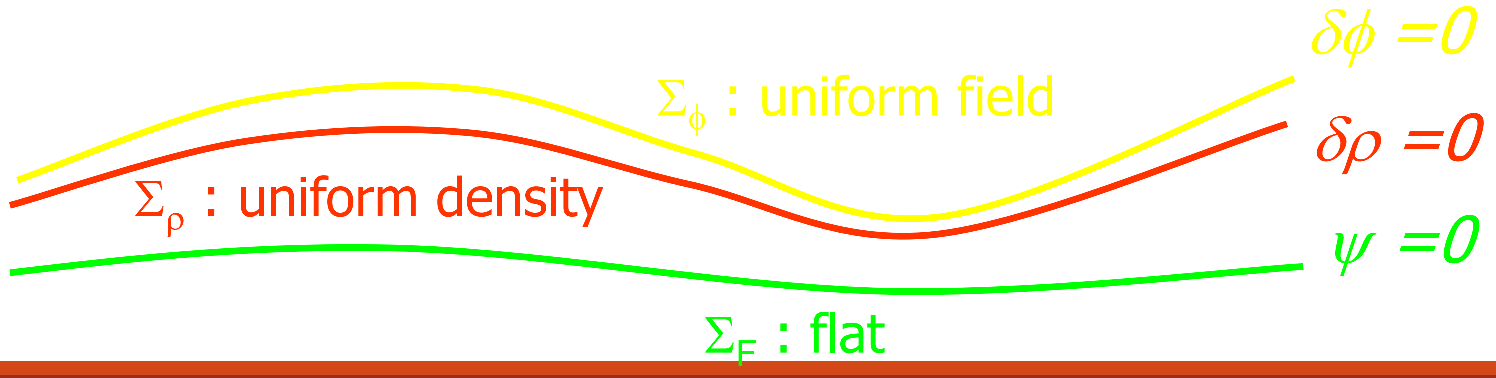


Gauge Invariants

- Combine two scalars $\widetilde{\psi}_1 = \psi_1 - \mathcal{H}\alpha_1\,, \;\; \widetilde{\delta\rho_1} = \delta\rho_1 + \rho_0'\alpha_1$

- Obtain a Gauge-invariant quantity $\frac{\psi_1}{\mathcal{H}} - \frac{\delta\rho_1}{\bar{\rho}'} = \frac{\tilde{\psi}}{\mathcal{H}} = -\frac{}{\mathcal{H}}$

The **Uniform density curvature**



Gauge Invariants

- Combine two scalars $\tilde{\psi}_1 = \psi_1 - \mathcal{H}\alpha_1 \,, \; \widetilde{\delta\rho_1} = \delta\rho_1 + \rho_0'\alpha_1$

- Obtain a Gauge-invariant quantity

The **Uniform density curvature** $\frac{\psi_1}{\mathcal{H}} - \frac{\delta\rho_1}{\bar{\rho}'} = \frac{\tilde{\psi}}{\mathcal{H}} = -\frac{}{\mathcal{H}}$

- **Bardeen Potentials** are the first

but not only gauge-invariants.

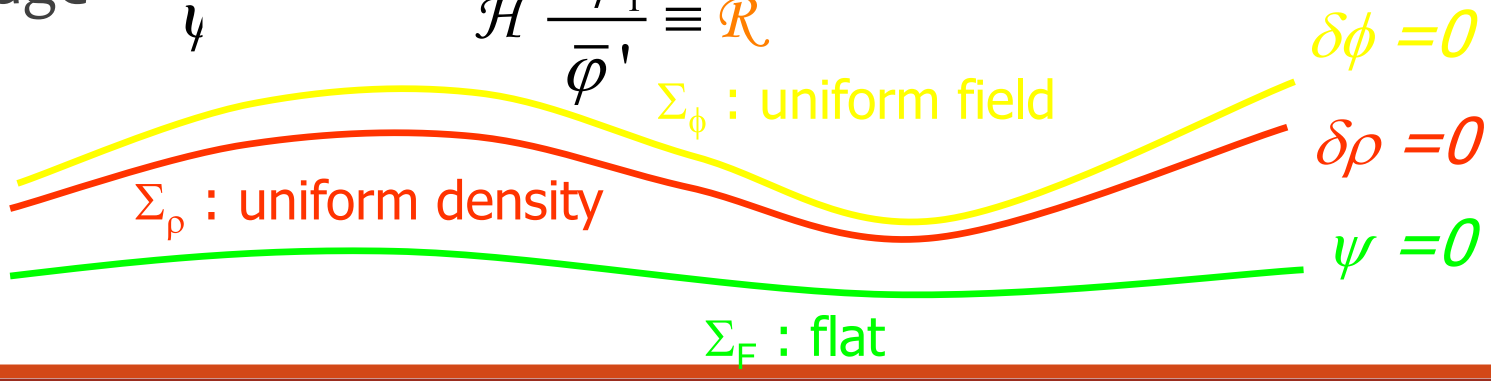
$$\tilde{\phi} \qquad \mathcal{H}\left(B_1 - E_1'\right) + \left(B_1 - E_1'\right)' \equiv \Phi$$

$$\tilde{\psi} \qquad -\mathcal{H}\left(B_1 - E_1'\right) \equiv \Psi$$

- Curvature Perturbation in

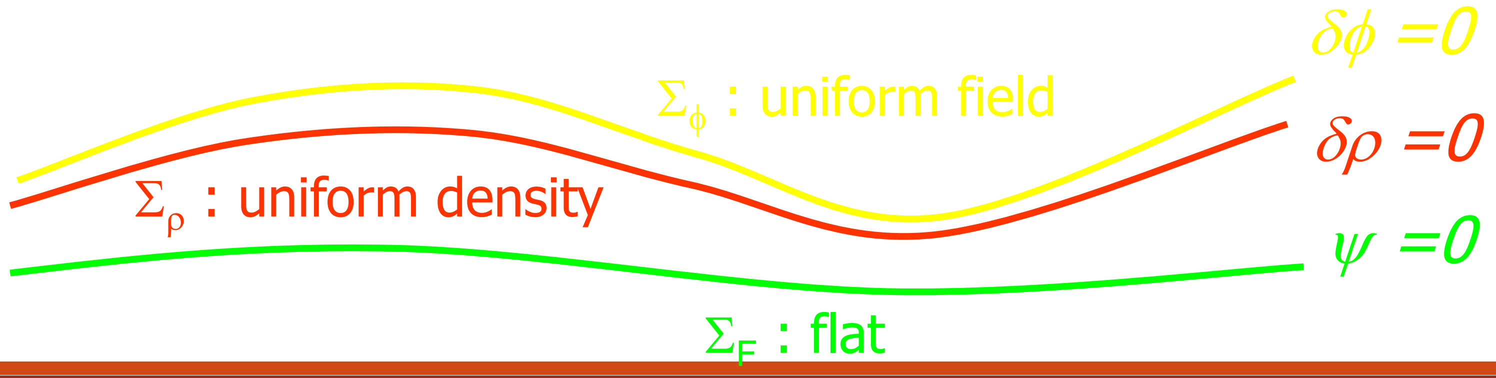
Uniform field (**comoving**) gauge

$$\tilde{\psi} \qquad \mathcal{H}\frac{\delta\varphi_1}{\bar{\varphi}'} \equiv \mathcal{R}$$



Gauge lessons

- Physically meaningful $\tilde{\zeta}$ are found by fixing a gauge completely.
- Gauge-invariant $\tilde{\zeta}$ is any fixed-gauge quantity.
- Gauge transformations show only two degrees of freedom $\xi^{\mu} = (\alpha, \beta^{i} + \beta_{,}^{\;i})$
- Different problems do with specific gauges.



Chapter 3. Perturbation Dynamics

Conservation Equations

- Energy conservation at first order (continuity equation)

$$\delta\rho' + 3\mathcal{H}\left(\delta\rho + \delta P\right) - 3\left(\rho + P\right)\psi' + (\rho + P)\nabla^2\left(V + \sigma\right) = 0\,,$$

- In terms of uniform density curvature (with $c_{\rm s}^2 \equiv \dfrac{P'}{\rho'}$):

$$\zeta' = -\mathcal{H}\frac{\delta P_{\rm nad}}{\rho + P} - \frac{1}{3}\nabla^2\tilde{v}_\ell$$

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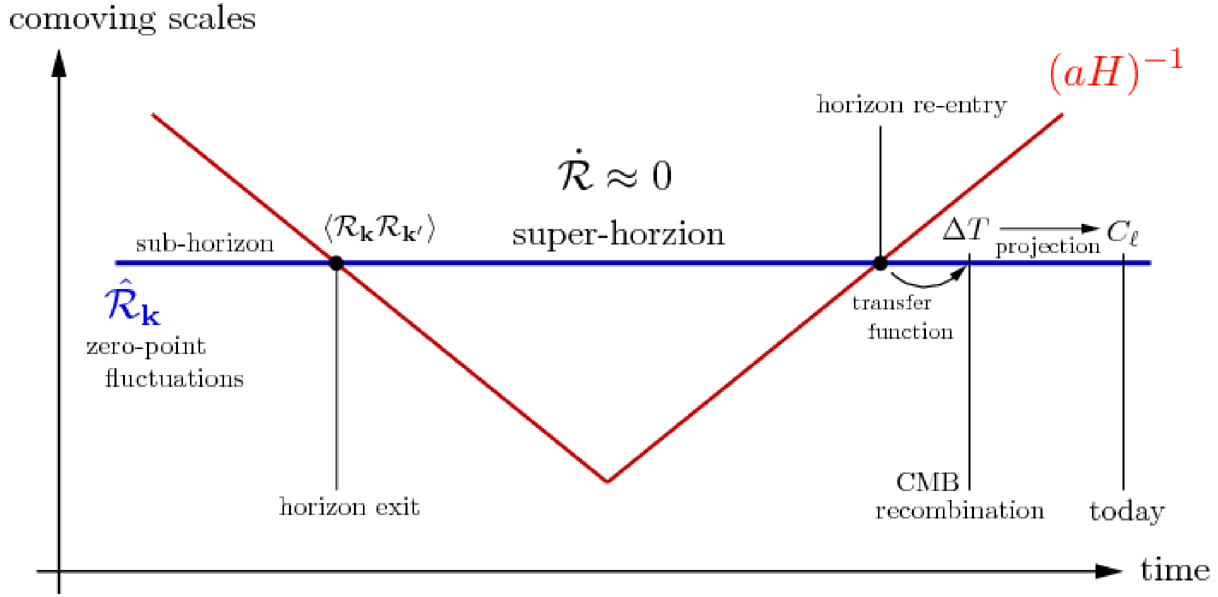
→ **Constant ζ for adiabatic δP_1 and large scales**

- Result valid at all orders, ζ is conserved if no entropy perturbations appear.



Conservation Equations

- Momentum conservation (Euler equation)

$$V' + (1 - 3c_{\rm s}^2)\mathcal{H}V + \phi + \frac{1}{\rho + P}\left(\delta P + \frac{2}{3}\nabla^2\Pi\right) = 0$$

Conservation Equations

- Momentum conservation (Euler equation)

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- In a comoving gauge: $V = v + B = 0$

$$(\rho + P)\phi = \delta P + (2/3)\nabla^2\Pi$$ $\longrightarrow$ **Acceleration produced by pressure gradients**

Conservation Equations

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- In a comoving gauge: $V = v + B = 0$

$$(\rho + P)\phi = \delta P + (2/3)\nabla^2\Pi$$ → **Acceleration produced by pressure gradients**

- For pressureless dust: $(aV)' + a\phi = 0$

→ **In synchronous dust velocity evolves as** $V_{\phi 1} \approx 1/a$

Conservation Equations

- Momentum conservation (Euler equation)

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- In a comoving gauge: $V = v + B = 0$

$$(\rho + P)\phi = \delta P + (2/3)\nabla^2\Pi$$ → **Acceleration produced by pressure gradients**

- For pressureless dust: $(aV)' + a\phi = 0$

→ **In synchronous dust velocity evolves as** $V_{\phi 1} \approx 1/a$

- In Longitudinal gauge, Euler + continuity:

[illegible]

¡Tarea!

Einstein Equations

- Energy and momentum constraints

$$3\mathcal{H}\left(\psi' + \mathcal{H}\phi\right) - \nabla^2\left[\psi + \mathcal{H}\sigma\right] = -4\pi G a^2 \delta\rho,$$

$$\psi' + \mathcal{H}\phi = -4\pi G a^2 (\rho + P) v + B$$

- In longitudinal gauge:

$$3\mathcal{H}\left(\Psi' + \mathcal{H}\Phi\right) - \nabla^2\Psi = -4\pi G a^2 \delta\rho_\ell,$$

$$\Psi' + \mathcal{H}\Phi = -4\pi G a^2 (\rho + P) v_\ell$$

$\rightarrow$

$$\boxed{\nabla^2\Psi = 4\pi G a^2 \delta\rho_{1\mathrm{com}}}$$

Poisson Equation at all scales!

Einstein Equations

- Evolution equations:

$$\psi'' + 2\mathcal{H}\psi' + \mathcal{H}\phi' + \left(2\mathcal{H}' + \mathcal{H}^2\right)\phi = 4\pi G a^2 \left(\delta P + \frac{2}{3}\nabla^2\Pi\right) ,$$

$$\sigma' + 2\mathcal{H}\sigma + \psi - \phi = 8\pi G a^2 \Pi$$

- In longitudinal gauge:

$$\Psi - \Phi = 8\pi G a^2 \Pi$$

→ **Equivalent potentials if no anisotropic stress**

$$\Psi'' + 3(1 + c_s^2)\mathcal{H}\Psi' + [2\mathcal{H}' + (1 + 3c_s^2)\mathcal{H}^2 - c_s^2\nabla^2]\Psi = 0$$

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- David Lyth, Andrew Liddle. The primordial density perturbation. CUP, 2009
- Karim Malik, David Wands. Cosmological perturbations arXiv:0809.4944v2
- David Langlois & Filippo Vernizzi.

Chapter 3. Perturbation Solutions

Chapter 4. Inflation Perturbations